Dual theories for mixed symmetry fields. Spin-two case: (1,1) versus (2,1) Young symmetry type fields.

H. Casini [a], R. Montemayor [b] and Luis F. Urrutia[c]

[a] *Theoretical Physics, University of Oxford, 1 Keble Road, Oxford OX13NP, United Kingdom*

[b] *Instituto Balseiro and CAB, Universidad Nacional de Cuyo and CNEA, 8400 Bariloche, Argentina*

[c] *Departamento de Física de Altas Energías, Instituto de Ciencias Nucleares Universidad Nacional Autónoma de México, A.P. 70-543, México D.F. 04510, México*

Abstract

We show that the parent Lagrangian method gives a natural generalization of the dual theories concept for non p-form fields. Using this generalization we construct here a three-parameter family of Lagrangians that are dual to the Fierz-Pauli description of a free massive spin-two system. The dual field is a three-index tensor $T_{(\mu\nu)\rho}$, which dinamically belongs to the $(2,1)$ representation of the Lorentz group. As expected, the massless limit of our Lagrangian, which is parameter independent, has two propagating degrees of freedom per space point.

I. INTRODUCTION

In general there is a lot freedom to choose variables for the description of a physical system. For example, in some cases it is desirable to have a formulation where some symmetries are explicit, and this requires the use of a redundant set of variables to describe the system configuration. In other cases, as when we perform a canonical quantization, it might be more convenient to have a minimal, non redundant, set of variables. The actual proof of the equivalence between different descriptions is usually a non-trivial task. Equivalent descriptions of a given physical theory in terms of a different choice of fields are said to be dual to each other, and the relation among the fields provides the corresponding duality transformation [1].

One of the simplest examples is the scalar-tensor duality. It corresponds to the equivalence between a free massless scalar field φ, with field strength $f_\mu = \partial_\mu \varphi$, and a massless antisymmetric field $B_{\mu\nu}$, the Kalb-Ramond field, with field strength $H_{\mu\nu\sigma} = \partial_\mu B_{\nu\sigma} + \partial_\nu B_{\sigma\mu} + \partial_\sigma B_{\mu\nu}$ [2–4]. The former description provides the equation of motion $\partial^\mu f_\mu = 0$, together with the Bianchi identities $\partial^\mu \epsilon_{\mu\nu\sigma}{}^{\tau} f_\tau = 0$. In turn, the dual description provides the equations of motion $\partial^\mu H_{\mu\nu\sigma} = 0$ and the Bianchi identities $\partial^\mu \epsilon_\mu{}^{\nu\sigma\tau} H_{\nu\sigma\tau} = 0$. The duality transformation

$$f_\mu \quad \leftrightarrow \quad \frac{1}{3}\epsilon_\mu{}^{\nu\sigma\tau} H_{\nu\sigma\tau} \tag{1}$$

interchanges dynamical equations with Bianchi identities, giving a full correspondence between both descriptions.

An important predecessor of the modern approach to duality is the electric-magnetic symmetry $(\vec{E}+i\vec{B}) \rightarrow e^{i\phi}(\vec{E}+i\vec{B})$ of the free Maxwell equations. When there are charged sources this symmetry can be maintained by introducing magnetic monopoles [5]. This transformation provides a connection between weak and strong couplings via the Dirac quantization condition. At the level of Yang-Mills theories with spontaneous symmetry breaking this kind of duality is expected, due to the existence of topological dyon-type solitons [6]. The extension of electromagnetic duality to $SL(2,Z)$ is usually referred to as S-duality, and plays an important role in the non-perturbative study of field and string theories [7].

These basic ideas have been subsequently generalized to arbitrary forms in arbitrary dimensions. Well known dualities are the ones between massless p -forms and $(d-p-2)$-forms fields and between massive p and $(d-p-1)$-forms in d dimensional space-time [8]. These dualities among free fields have been proved using parent Lagrangians [9] as well as the canonical formalism [10]. They can be extended to include source interactions [11].

The above duality among forms can be understood as a relation between fields in different representations of the Lorentz group. The origin of this equivalence can be traced using the little group technique for constructing the representations of the Poincare group in d dimensions. Given a standard momentum for a massive or a massless particle, the actual degrees of freedom are determined by its spin components, which are given by the irreducible representations of $SO(d-1)$ and $SO(d-2)$ respectively. These are expressed by traceless tensors with a definite permutation symmetry characterized by the Young diagrams. For the orthogonal groups $O(n)$, the sum of the lengths of the first two columns of the Young diagrams is constrained to be less than or equal to n. Two Young diagrams having their first column of length l and $n-l$ respectively, with the rest of the diagrams being identical, are called associated. For the group $SO(n)$ associated Young diagrams correspond to the same representation [12]. The existence of two different tensorial realizations for one irreducible spin representation suggests that we can construct two theories with fields of different tensorial character for the same physical system. The relation between two equivalent theories expressed in terms of fields corresponding to associated Young diagrams should be interpreted as a duality transformation.

In particular, the usual duality among p-forms can be interpreted in this way. A (traceless) p-form belongs to a Young diagram with only one column and p sites, and a $(d-p-1)$-form belongs to a Young diagram with one column and $(d-p-1)$ sites. Thus they give the same representation of $SO(d-1)$ and have the same spin; therefore, if the masses are equal there is a possibility of constructing alternative theories for the same physical system. The same can be said regarding the massless case if we consider p and $(d-p-2)$ forms.

The preceding discussion suggests the possibility of generalizing the duality transformations among p-forms to tensorial fields with arbitrary Young symmetry types. The simplest generalization in four dimensions is provided by a massive second-rank symmetric tensor $h_{\mu\nu}$, with Young symmetry $(1,1)$, and a three index tensor $T_{\mu\nu\rho}$ with Young symmetry $(2,1)$. Consistent massless free [13], interacting [14], and massive [15] theories of mixed Young symmetry tensors have been constructed in the past, but the attempts to prove a dual relation between these descriptions did not lead to a positive answer [15]. Additional

interest in this type of theories arises from the recent formulation of $d = 11$ dimensional supergravity as a gauge theory for the osp(32|1) superalgebra. It includes a totally antisymmetric fifth-rank Lorentz tensor one form $b_\mu{}^{abcde}$, whose mixed symmetry piece does not have any related counterpart in the standard $d = 11$ supergravity theory [16].

In this work we show that such dual descriptions can be constructed. We use a generalization of the parent Lagrangian method to construct a family of Lagrangians for the massive field $T_{(\mu\nu)\rho}$, which are dual to the standard Fierz-Pauli Lagrangian for a massive spin two field $h_{\mu\nu}$. In contrast with the p-forms case, we obtain a multiparameter family of duality transformations. Most notably, the kinetic part of the dual T-Lagrangians is unique, and the parameters appear only in the mass term. The massless limit of these descriptions is well-behaved in the sense that it has the correct two propagating degrees of freedom, in contrast with the absence of degrees of freedom found in previous attempts [15].

II. MASSIVE SPIN-TWO IRREDUCIBLE REPRESENTATIONS

In the following, we perform a detailed construction for the massive spin-two fields in four dimensions. There are two associated Young diagrams, which correspond to a traceless symmetric rank-two tensor, and to a traceless rank-three tensor, antisymmetric in two indices, and satisfying a Jacobi identity. The most usual known description is given in terms of a symmetric tensor $h_{\mu\nu} = h_{\nu\mu}$, with the dynamics defined by the Fierz-Pauli Lagrangian

$$\mathcal{L} = -\partial_\mu h^{\mu\nu}\partial_\alpha h^\alpha_\nu + \frac{1}{2}\partial_\alpha h^{\mu\nu}\partial^\alpha h_{\mu\nu} + \partial_\mu h^{\mu\nu}\partial_\nu h^\alpha_\alpha - \frac{1}{2}\partial_\alpha h^\mu_\mu \partial^\alpha h^\nu_\nu - \frac{M^2}{2}\left(h_{\mu\nu}h^{\mu\nu} - h^\mu_\mu h^\nu_\nu\right). \quad (2)$$

¿From the equation of motion it follows that

$$\left(\partial^2 + M^2\right) h_{\mu\nu} = 0, \qquad \partial_\mu h^{\mu\nu} = 0, \qquad h_\mu{}^\mu = 0, \quad (3)$$

which leads to the five degrees of freedom corresponding to a massive spin-two irreducible representation.

An alternative description for the free massive spin-two field corresponding to the associated $(2, 1)$ Young diagram has been already proposed [15]. It is based on the tensor field $T_{(\mu\nu)\sigma}$ that satisfies

$$T_{(\mu\nu)\sigma} = -T_{(\nu\mu)\sigma} \quad ; \quad T_{(\mu\nu)\sigma} + T_{(\nu\sigma)\mu} + T_{(\sigma\mu)\nu} = 0. \quad (4)$$

At this stage $T_{(\mu\nu)\sigma}$ has 20 independent components. The proposed Lagrangian is

$$\mathcal{L} = -\frac{1}{36}\left\{\left(F_{(\mu\nu\sigma)\tau}\right)^2 - 3\left(F_{(\mu\nu\sigma)}{}^\mu\right)^2 - 3M^2\left[\left(T_{(\mu\nu)\sigma}\right)^2 - 2\left(T_{(\mu\nu)}{}^\mu\right)^2\right]\right\} \quad (5)$$

with the field strength given by

$$F_{(\mu\nu\sigma)\tau} = \partial_\mu T_{(\nu\sigma)\tau} + \partial_\nu T_{(\sigma\mu)\tau} + \partial_\sigma T_{(\mu\nu)\tau}. \quad (6)$$

In this case the equations of motion imply:

$$\left(\partial^2 + M^2\right) T_{(\mu\nu)\sigma} = 0, \quad T_{(\mu\nu)}{}^\mu = 0, \quad \partial^\mu T_{(\mu\nu)\sigma} = 0, \quad \partial^\sigma T_{(\mu\nu)\sigma} = 0. \quad (7)$$

The first algebraic condition gives four identities. The second derivative condition is a consequence of the first one plus the cyclic identity in (4). The first derivative condition includes four identities, $\partial^\mu \partial^\nu T_{\mu\nu\rho} = 0$, plus one more, $\partial^\mu T_{\mu\rho}{}^\rho$, when we consider independently the zero-trace condition. This leads to 11 independent derivative conditions. The final count produces 5 independent degrees of freedom, as appropriate to a massive spin-two field. It is natural to suspect that both theories could be related by a duality transformation, but the attempts to construct such a transformation have had no success [15]. As we will show below the Lagrangian (5) is only one of the possible descriptions for an irreducible massive spin two field in terms of a $T_{(\mu\nu)\sigma}$ tensor, and in fact is not a suitable choice to construct a dual transformation.

III. PARENT LAGRANGIAN FOR DUAL THEORIES

A standard procedure to construct alternative descriptions for a physical system, which turn out to be related by a duality transformation, consists in finding a quadratical parent Lagrangian that contains both types of fields, from which each theory can be obtained by eliminating either one of them through the corresponding equations of motion. In this work we show that a generalization of this method, already successfully applied to p-forms, allows us to construct dual theories for the massive spin-two representation. Contrary to the approach in [15], we start from a tensor $T_{(\mu\nu)\sigma}$ which is only antisymmetric in the $\mu\nu$ indices, without a priori satisfying the cyclic identity, together with the standard symmetric tensor $h_{\mu\nu}$. Eliminating the field $T_{(\mu\nu)\sigma}$ from the parent Lagrangian we impose the resulting Lagrangian for $h_{\mu\nu}$ to be the Fierz-Pauli one describing a massive spin-two system. This provides relations among the parameters of the model. Once these relations are implemented in the parent Lagrangian, the elimination of field $h_{\mu\nu}$ leads to the required dual Lagrangian in terms of field $T_{(\mu\nu)\sigma}$. The necessary Lagrangian constraints leading to the $(2,1)$ Young symmetry type of the field together with the required five independent degrees of freedom come from the corresponding Euler-Lagrangian equations.

Let us recall the general structure of the parent Lagrangian used to establish duality between a massive $d-q-1$ form L and a massive q form B

$$\mathcal{L}_P = \frac{1}{2} L \wedge {}^*L + L \wedge dB + \frac{M^2}{2} B \wedge {}^*B, \tag{8}$$

which we will generalize to our case. The most general first order bilinear Lagrangian for $T_{(\mu\nu)\sigma}$ and $h_{\mu\nu}$ has seven bilinears in T

$$\begin{gathered} T_{(\mu\nu)\sigma} T^{(\mu\nu)\sigma}, \quad T^{\nu}_{(\mu\nu)} T^{(\mu\sigma)}_{\sigma}, \quad T_{(\mu\nu)\sigma} T^{(\mu\sigma)\nu}, \\ \epsilon^{\mu\nu\alpha\beta} T_{(\mu\nu)\sigma} T^{\sigma}_{(\alpha\beta)}\,, \quad \epsilon^{\mu\nu\alpha\beta} T_{(\mu\nu)\sigma} T^{(\sigma}{}_{\alpha)\beta}\,, \quad \epsilon^{\mu\nu\alpha\beta} T_{(\mu\sigma)\nu} T^{(\sigma}{}_{\alpha)\beta}\,, \quad \epsilon^{\mu\nu\alpha\beta} T_{(\mu\nu)\alpha} T^{\lambda}_{(\beta\lambda)}, \end{gathered} \tag{9}$$

two in h,

$$h_{\mu\nu} h^{\mu\nu}, \quad h^{\mu}_{\mu} h^{\nu}_{\nu}, \tag{10}$$

and seven mixing (duality generating) terms which contain both fields,

$$T_{(\mu\nu)\sigma}\partial^\mu h^{\nu\sigma}\,, \quad T_{(\mu\nu)}{}^{\nu}\partial^\mu h^\sigma_\sigma\,, \quad T_{(\mu\nu)}{}^{\nu}\partial^\sigma h^\mu_\sigma,$$
$$T_{(\mu\nu)\sigma}\epsilon^{\mu\nu\alpha\beta}\partial_\alpha h^\sigma{}_\beta, \quad T_{(\mu\nu)\sigma}\epsilon^{\mu\nu\sigma\beta}\partial_\alpha h^\alpha{}_\beta\,, \quad T_{(\mu\nu)\sigma}\epsilon^{\mu\nu\sigma\beta}\partial_\beta h^\alpha{}_\alpha\,, \quad T_{(\mu\nu)\sigma}\epsilon^{\mu\sigma\alpha\beta}\partial_\beta h^\nu{}_\alpha\,. \tag{11}$$

In the following, and for the sake of simplicity, we will explore in detail only the Lagrangian with one mixing term, $T_{(\mu\nu)\sigma}\epsilon^{\mu\nu\alpha\beta}\partial_\alpha h^\sigma{}_\beta$, because this is the most natural generalization of the term $L \wedge d\,B$ in (8). Thus, we take our parent Lagrangian to be

$$\begin{aligned} L = {} & a\,T_{(\mu\nu)\sigma}T^{(\mu\nu)\sigma} + b\,T_{(\mu\nu)}{}^{\nu}T^{(\mu\sigma)}{}_{\sigma} + c\,T_{(\mu\nu)\sigma}T^{(\mu\sigma)\nu} + d\,\epsilon^{\mu\nu\alpha\beta}T_{(\mu\nu)\sigma}T_{(\alpha\beta)}{}^{\sigma} \\ & + m\,\epsilon^{\mu\nu\alpha\beta}T_{(\mu\nu)}{}^{\sigma}T{}_{(\sigma\alpha)\beta} + n\,\epsilon^{\mu\nu\alpha\beta}T_{(\mu\sigma)\nu}T^{(\sigma}{}_{\alpha)\beta} + p\epsilon^{\mu\nu\alpha\beta}T_{(\mu\nu)\alpha}T_{(\beta\lambda)}{}^{\lambda} \\ & + eT_{(\mu\nu)\sigma}\epsilon^{\mu\nu\alpha\beta}\partial_\alpha h^\sigma{}_\beta \\ & + f h_{\mu\nu}h^{\mu\nu} + k h_\mu{}^\mu h_\nu{}^\nu . \end{aligned} \tag{12}$$

The equations of motion for $h_{\mu\nu}$ allow us to solve algebraically this field in terms of $T_{(\mu\nu)\sigma}$

$$h^{\mu\nu} = \frac{e}{4f}\left(\epsilon^{\alpha\beta\sigma\nu}\partial_\sigma T_{(\alpha\beta)}{}^{\mu} + \epsilon^{\alpha\beta\sigma\mu}\partial_\sigma T_{(\alpha\beta)}{}^{\nu}\right) - g^{\mu\nu}\frac{ke}{2\,(f+4k)\,f}\epsilon^{\alpha\beta\rho\kappa}\partial_\rho T_{(\alpha\beta)\kappa}. \tag{13}$$

In a similar way, from the Euler-Lagrange equation for $T^{(\mu\nu)\sigma}$ we can algebraically solve this field in terms of $h^{\mu\nu}$

$$C\,T^{(\kappa\lambda)\sigma} = -e\,(2m+n-4d)\,D^{\lambda\sigma\kappa} + \frac{e}{2}\,(2a+c)\,\epsilon_{\mu\nu}{}^{\kappa\lambda}D^{\nu\sigma\mu} - \frac{e}{2}E\,\epsilon_\mu{}^{\sigma\kappa\lambda}D^\mu - \frac{e}{2}F\,G^{\sigma\kappa\lambda}\,. \tag{14}$$

where

$$\begin{aligned} & D^{\lambda\sigma\kappa} \equiv \partial^\lambda h^{\sigma\kappa} - \partial^\kappa h^{\sigma\lambda}, \qquad D^\mu \equiv \partial_\beta h^{\beta\mu} - \partial^\mu h^\beta_\beta = D_\sigma{}^{\sigma\mu}, \\ & G^{\sigma\kappa\lambda} \equiv g^{\sigma\kappa}\left(\partial_\alpha h^{\alpha\lambda} - \partial^\lambda h^\alpha_\alpha\right) - g^{\sigma\lambda}\left(\partial_\alpha h^{\alpha\kappa} - \partial^\kappa h^\alpha_\alpha\right) = g^{\sigma\kappa}D^\lambda - g^{\sigma\lambda}D^\kappa \end{aligned} \tag{15}$$

and

$$\begin{aligned} & A = (4d+m-3p+2n)\left[(4d+m-3p+2n)^2 + 2(2a+c+3b)\,(a-c)\right]^{-1}, \\ & B = -2(2a+c+3b)\left[(4d+m-3p+2n)^2 + 2(2a+c+3b)\,(a-c)\right]^{-1}, \\ & C = (2m+n-4d)^2 + (2a+c)^2\,, \\ & E = (2m+n-4d)\,(2bA + (m+n-p)B) + (2a+c)\,(2(m+n-p)A + cB)\,, \\ & F = (2m+n-4d)\,(2(m+n-p)A + cB) + (2a+c)\,(2bA + (m+n-p)B)\,. \end{aligned} \tag{16}$$

The parent Lagrangian we are considering contains seven bilinears in T, two bilinears in h, and one mixing term, which means we have started with ten parameters. Nevertheless, only eight combinations appear in the dual transformations (13) and (14): a, b, c, $(m+n-p)$, $(2m+n-4d)$, e, f, and k. Note that $(4d+m-3p+2n) = 3(m+n-p) - (2m+n-4d)$. We further fix appropriate combinations by using Eq. (14) to rewrite the parent Lagrangian (12) in terms of $h^{\mu\nu}$ only, and subsequently demanding it to be the Fierz-Pauli Lagrangian for a massive spin two field. The result is

$$C = 2e^2\,(2a+c)\,, \quad E = 2\,(2a+c)\,, \quad f = -k = -\frac{M^2}{2}, \tag{17}$$

The parameters d, m, n and p only appear through the combinations $(2m+n-4d)$ and $(m+n-p)$, each of which can be written in terms of a, b, c and e using Eqs. (17). Considering that the same equations of motion are obtained up to an overall factor in the Lagrangian, we have constructed a three-parameter family of parent Lagrangians for a spin-two field with mass M leading to the Fierz-Pauli Lagrangian for $h_{\mu\nu}$.

IV. SPIN-TWO MIXED SYMMETRY DESCRIPTION

Now we consider the dual description of the massive spin two system in terms of the field $T^{(\mu\nu)\rho}$. To this end we use Eq. (13) and rewrite the parent Lagrangian as

$$\begin{aligned} L = & -\frac{e^2}{3M^2}\left(\left(F_{(\mu\nu\sigma)\tau}\right)^2 - \frac{3}{2}\left(F_{(\mu\nu\sigma)}{}^{\mu}\right)^2 + \frac{1}{2}\left(\epsilon^{\mu\nu\alpha\beta}\,\partial_\beta T_{(\mu\nu)\alpha}\right)^2\right) \\ & + a\,T_{(\mu\nu)\sigma}T^{(\mu\nu)\sigma} + b\,T_{(\mu\nu)}{}^{\nu}T^{(\mu\sigma)}{}_{\sigma} + c\,T_{(\mu\nu)\sigma}T^{(\mu\sigma)\nu} + d\,\epsilon^{\mu\nu\alpha\beta}\,T_{(\mu\nu)\sigma}T_{(\alpha\beta)}{}^{\sigma} \\ & + m\,\epsilon^{\mu\nu\alpha\beta}\,T_{(\mu\nu)}{}^{\sigma}T_{(\sigma\alpha)\beta} + n\,\epsilon^{\mu\nu\alpha\beta}\,T_{(\mu\sigma)\nu}T^{(\sigma}{}_{\alpha)\beta} + p\,\epsilon^{\mu\nu\alpha\beta}\,T_{(\mu\nu)\alpha}T_{(\beta\lambda)}{}^{\lambda}. \end{aligned} \quad (18)$$

The second term of the kinetic part of this Lagrangian has a coefficient $\left(-\frac{3}{2}\right)$ instead of the corresponding (-3) in the Lagrangian (5). This fact was recognized in [15] as a potential problem for dualization. However, here we do not impose the cyclic property (4) for $T_{(\mu\nu)\sigma}$ from the beginning, and this is manifested in the existence of the third term in the kinetic Lagrangian, which is crucial for dualization.

It turns out that the trace $T_\mu \equiv T_{(\mu\beta)}{}^{\beta}$ is an auxiliary field in the above Lagrangian. This is made explicit by defining

$$T_{(\mu\nu)\sigma} = \hat{T}_{(\mu\nu)\sigma} - \frac{1}{3}\left(g_{\mu\sigma}T_\nu - g_{\nu\sigma}T_\mu\right), \quad (19)$$

where $\hat{T}_{(\mu\nu)\sigma}$ is a traceless field, $\hat{T}_{(\mu\nu)}{}^{\nu} = 0$. Rewriting the Lagrangian (18) in terms of $\hat{T}^{(\chi\psi)\sigma}$ and T_μ and using the Euler-Lagrangian equation for T_μ we obtain

$$T^\beta = \frac{4d + m + 2n - 3p}{2\left(2a + 3b + c\right)}\epsilon^{\mu\nu\alpha\beta}\,\hat{T}_{(\mu\nu)\alpha}, \quad (20)$$

which allows us to algebraically eliminate the trace in the Lagrangian (18). We further introduce the field strength $\hat{F}_{(\alpha\beta\gamma)\nu}$ appropriate to $\hat{T}^{(\mu\nu)\sigma}$, defined in (6). In this way, up to a global factor, (18) can be written as

$$\begin{aligned} L = & \frac{4}{9}\hat{F}_{(\alpha\beta\gamma)\nu}\,\hat{F}^{(\alpha\beta\gamma)\nu} + \frac{2}{3}\hat{F}_{(\alpha\beta\gamma)\nu}\,\hat{F}^{(\alpha\beta\nu)\gamma} - \hat{F}_{(\alpha\beta\mu)}{}^{\mu}\,\hat{F}^{(\alpha\beta\nu)}{}_{\nu} \\ & + P\,\hat{T}_{(\nu\rho)\sigma}\hat{T}^{(\nu\rho)\sigma} + Q\,\hat{T}_{(\nu\rho)\sigma}\hat{T}^{(\nu\sigma)\rho} + R\,\epsilon^{\mu\nu\alpha\beta}\,\hat{T}_{(\mu\nu)\sigma}\hat{T}_{(\alpha\beta)}{}^{\sigma} + \lambda^\mu\hat{T}_{(\mu\beta)}{}^{\beta}, \end{aligned} \quad (21)$$

where we have introduced the Lagrange multiplier λ^μ to enforce the traceless condition upon $\hat{T}_{(\mu\sigma)\nu}$. Here it is

$$P = \frac{2M^2}{3e^2}\left(\frac{1}{B} - (2a + c)\right), \quad Q = -\frac{2M^2}{3e^2}\left(\frac{2}{B} + (2a + c)\right), \quad R = \frac{M^2}{2e^2}\left(2m + n - 4d\right). \quad (22)$$

The equations (17) translate in the following relations between these parameters

$$\begin{aligned} & 12\,P^2 - 3\,Q^2 + 16\,R^2 = 0, \\ & \frac{1}{4}(2P + Q) + 4\frac{R^2}{(2P + Q)} = -M^2. \end{aligned} \quad (23)$$

This family of Lagrangians, dual to the Fierz-Pauli Lagrangian (2) by construction, is the main result of this work.

Let us recall that in (21) the field $\hat{T}_{(\mu\nu)\sigma}$ is only antisymmetric and traceless, therefore having 20 components, but the Euler-Lagrange equations give rise to the necessary Lagrangian constraints leading to the final required five independent degrees of freedom. The final equations of motion together with the independent Lagrangian constraints which completely determine the dynamics are

$$\begin{aligned} \partial_\alpha \left(\partial^\alpha \hat{T}^{(\nu\rho)\sigma} + \partial^\nu \hat{T}^{(\rho\alpha)\sigma} + \partial^\rho \hat{T}^{(\alpha\nu)\sigma}\right) - \frac{1}{4}\left(Q+2P\right)\hat{T}^{(\nu\rho)\sigma} - \frac{1}{2} R\, \epsilon^{\nu\rho\alpha\beta} \hat{T}_{(\alpha\beta)}{}^{\sigma} = 0, \\ \hat{T}_{(\mu\beta)}{}^{\beta} = 0, \quad \epsilon_{\alpha\beta\gamma\lambda}\,\hat{T}^{(\beta\gamma)\lambda} = 0, \quad \partial_\sigma \hat{T}_{(\alpha\beta)}{}^{\sigma} = 0, \\ \partial_\nu \hat{T}^{(\nu\rho)\sigma} + \frac{R}{2P+Q}\left(\epsilon^{\nu\rho\alpha\beta}\,\partial_\nu \hat{T}_{(\alpha\beta)}{}^{\sigma} + \epsilon^{\nu\sigma\alpha\beta}\,\partial_\nu \hat{T}_{(\alpha\beta)}{}^{\rho}\right) = 0. \end{aligned} \tag{24}$$

To be sure we have the correct number of degrees of freedom and to identify the mass parameter it is convenient to use plane wave solutions in a rest frame, where $k^\mu = (\mu, \mathbf{0})$. In this frame the equations of motion read

$$\left(\frac{1}{4}(Q+2P) + \mu^2\right)\hat{T}^{(\nu\rho)\sigma} + \mu^2\left(g^{0\nu}\hat{T}^{(\rho 0)\sigma} + g^{0\rho}\hat{T}^{(0\nu)\sigma}\right) + \frac{1}{2} R\, \epsilon^{\nu\rho\alpha\beta}\hat{T}_{(\alpha\beta)}^{\ \ \sigma} = 0 \tag{25}$$

with the constraints

$$\hat{T}_{(\alpha\beta)}{}^{0} = 0, \quad \hat{T}^{(0i)k} = -\frac{R}{2P+Q}\left(\epsilon^{imn}\,\hat{T}_{(mn)}{}^{k} + \epsilon^{kmn}\,\hat{T}_{(mn)}{}^{i}\right). \tag{26}$$

Therefore, all the degrees of freedom can be expressed in terms of the nine components $\hat{T}_{(ij)k}$, which satisfy the four constraints

$$\hat{T}_{(ki)}{}^{i} = 0, \qquad \epsilon_{ijk}\,\hat{T}^{(ij)k} = 0. \tag{27}$$

and hence, as expected, five degrees of freedom remain. Using these constraints the equations of motion become

$$\left(\frac{1}{4}(Q+2P) + 4\frac{R^2}{Q+2P} + \mu^2\right)\hat{T}^{(jk)l} = 0, \tag{28}$$

which using (23) gives the mass $\mu = M$ for the $\hat{T}^{(jk)l}$ field.

The divergence $\partial_\kappa T^{(\kappa\lambda)\sigma}$ is non null if $(2m+n-4d) \neq 0$. From Eq. (14) the on-shell duality relation can be written

$$T^{(\kappa\lambda)\sigma} = eC^{-1}\left(-\left(2m+n-4d\right)\left(\partial^\lambda h^{\sigma\kappa} - \partial^\kappa h^{\sigma\lambda}\right) + (2a+c)\,\epsilon_{\mu\nu}^{\ \ \kappa\lambda}\partial^\nu h^{\sigma\mu}\right), \tag{29}$$

and we see that the divergence is proportional to $h^{\sigma\lambda}$ on the equations of motion

$$\partial_\kappa T^{(\kappa\lambda)\sigma} = -eC^{-1}M^2\left(2m+n-4d\right)h^{\sigma\lambda}. \tag{30}$$

The linear combination of the tensors appearing on the right hand side of (29) is the most general form that can take the tensor $T^{(\mu\nu)\sigma}$ of symmetry $(2,1)$ as a function of $h^{\sigma\kappa}$ on the equations of motion. Taking $(2m+n-4d) = 0$, the second term on the right hand side gives the standard form of the duality involving the Levi-Civita tensor. For null spatial momentum

this becomes $T^{(ij)k} \sim \epsilon_l{}^{ij} h^{lk}$. This is the duality, in the sense of representations of $SO(n)$, for the tensors of types $(1,1)$ and $(2,1)$ in three spatial dimentions. On the other hand the term proportional to $\left(\partial^\lambda h^{\sigma\kappa} - \partial^\kappa h^{\sigma\lambda}\right)$ makes the symmetric part of the divergence $\partial_\kappa T^{(\kappa\mu)\nu}$ non null and proportional to $h_{\mu\nu}$. It corresponds to giving $T^{(\mu\nu)\sigma}$ some longitudinal components, as can be seen in the zero spatial momentum where $T^{(0i)j} \sim h^{ij}$. Thus, the physical degrees of freedom of $h_{\mu\nu}$ are mapped either on the null divergence part of $T^{(\mu\nu)\sigma}$ or on the symmetric part of the divergence $\partial_\kappa T^{(\kappa\mu)\nu}$, and in the general case on a linear combination of both. The case where $h_{\mu\nu}$ is completely mapped on the divergence of $T^{(\mu\nu)\sigma}$ can not be achieved with the Lagrangian (12) because it would require $(2a+c)=0$, in which case the transformation becomes ill defined, but can be realized including the aditional duality term $T_{(\mu\nu)\sigma}\partial^\mu h^{\nu\sigma}$. We note that although all the parent Lagrangians with three free parameters represent the same theory, they give place to different duality mappings, characterized by one parameter.

We can see that in the parity-conserving case $2m+n-4d=0\,(R=0)$, the equations of motion and the Lagrangian constraints simplify to

$$\left(\partial^2 + M^2\right)\hat{T}^{(\nu\rho)\sigma} = 0, \quad \hat{T}^{(\nu\rho)}{}_\rho = 0, \quad \epsilon_{\kappa\nu\rho\sigma}\hat{T}^{(\nu\rho)\sigma} = 0, \quad \partial_\alpha \hat{T}^{(\alpha\nu)\sigma} = 0, \quad \partial_\alpha \hat{T}^{(\nu\rho)\alpha} = 0. \tag{31}$$

V. MASSLESS SPIN-TWO DUAL THEORIES

Here we will include only a few comments in relation with the zero-mass case, as a detailed discussion is postponed for a separate publication. To count the independent degrees of freedom we can use the Hamiltonian analysis of the zero-mass limit $(P=Q=R=0)$ of the Lagrangian (21). The $(3+1)$ splitting of degrees of freedom produces the coordinates $\hat{T}^{i00}$, $\hat{T}^{0ij}$, $\hat{T}^{ij0}$, $\hat{T}^{ijk}$ together with their corresponding canonically conjugated momenta Π_{i00}, Π_{0ij}, Π_{ij0}, Π_{ijk}, satisfying the standard equal-time non-zero Poisson brackets. The whole set of constraints, which include up to terciary constraints, is

$$\begin{aligned}
&\Gamma^i = \Pi^{i00}, \quad \Gamma^{ij} = \Pi^{0ij}, \quad \Gamma = \Pi - 4F^0, \quad \Delta = \hat{T}^{0m}{}_m, \quad \Delta^i = \hat{T}^{i00} + \hat{T}^{im}{}_m,\\
&\Sigma = \epsilon^{ijk}\partial_k \Pi_{ij0}, \quad \Sigma^{ij} = \partial_m \Pi^{mij} - \frac{1}{3} g^{ij}\partial_m \pi^m, \qquad (\Sigma^{ij} g_{ij} = 0),\\
&\Phi_{jm} = \partial_m \partial^q \Pi_{jq0}, \qquad (\partial^j \Phi_{jm} = 0, \quad g^{jm}\Phi_{jm} = 0),
\end{aligned} \tag{32}$$

with $\Pi = -\frac{1}{2}\,\epsilon_{ijk}\,\Pi^{ijk}$, $F^0 = -\frac{1}{2}\,\epsilon_{ijk}\,\partial^i\,\hat{T}^{jk0}$, $\pi^m = \Pi^{mk}{}_k$. In parenthesis we have indicated the identities satisfied by the corresponding constraints. Due to this we have 31 independent constraints. The first class constraints turn out to be the $9-1=8$ traceless components of Γ^{ij} together with the $9-3-1=5$ independent components of Φ^{jm}. Thus, 13 constraints are first class, while the remaining 18 are second class. This gives $\frac{1}{2}(2\times 24 - 2\times 13 - 18) = 2$ independent degrees of freedom in the configuration space.

Finally, a comment on the possible dual systems that could arise in the massless case. Let us recall the situation in the case of p-forms. Basically there are two ways of taking the zero-mass limit. One arises directly from the Lagrangian (8) and gives

$$\mathcal{L}_P = \frac{1}{2} L \wedge {}^*L + L \wedge dB. \tag{33}$$

The second possibility is to rescale the forms $B \rightarrow M^{-1}\; B$, $L \rightarrow ML$ in (8), which leads to the alternative massive Lagrangian

$$\mathcal{L}_P = \frac{M^2}{2} L \wedge {}^*L + L \wedge dB + \frac{1}{2} B \wedge {}^*B, \tag{34}$$

together with the corresponding zero mass limit

$$\mathcal{L}_P = L \wedge dB + \frac{1}{2} B \wedge {}^*B. \tag{35}$$

In the case of $M \neq 0$ the theories generated by the Lagrangians (8) and (34) are equivalent. Nevertheless, for $M = 0$ the situation is different. In fact, the Lagrangian (33) gives the equation of motion $dL = 0 \Rightarrow L = dA$, where A is a $(d-q-2)$-form, thus producing a duality of the type $q \Leftrightarrow (d-q-2)$. In an analogous way, the Lagrangian (35) produces the equation of motion $dB = 0$, thus introducing a $(q-1)$-form C such that $B = dC$. Here the duality is of the type $(q-1) \Leftrightarrow (d-q-1)$. Both types of dualities are not equivalent. The two possible dualities mentioned above arise from the mixing term $L \wedge d\, B$, plus the choice of the auxiliary field. In our case, the corresponding mixing term is

$$e\, T_{(\mu\nu)\sigma} \epsilon^{\mu\nu\alpha\beta} \partial_\alpha h^\sigma{}_\beta, \tag{36}$$

which gives the alternative field equations

$$\partial_\alpha \left(\epsilon^{\mu\nu\alpha\beta} T_{(\mu\nu)}{}^\sigma + \epsilon^{\mu\nu\alpha\sigma} T_{(\mu\nu)}{}^\beta \right) = 0, \qquad \epsilon^{\mu\nu\alpha\beta} \partial_\alpha h_{\sigma\beta} = 0. \tag{37}$$

Let us emphasize that they do not correspond to a condition of the form $dB = 0$, whose solution is given in terms of the Poincare lemma. Recent generalizations of the Poincare lemma for mixed symmetry tensors [17] could be useful to find the general solutions of the equations above, leading to an explicit construction of the corresponding dual systems.

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